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METHOD FOR FILTERING METEOROLOGICAL DATA

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ABSTRACT

A mathematical filter for eliminating persistence in meteorological data is proposed and discussed. This filter takes the form $Z_t = X_t - \rho_1 X_{t-1}$. Relationships between statistical parameters of the filtered and the original data are derived and found to depend only on the value of ρ_1. Examples of the effect of the filter on the power spectrum of various types of input data are also given.

1. INTRODUCTION

Long recognized is the fact that the analysis of meteorological data often requires the use of statistical formulas under conditions other than those assumed in their derivations. The generally limited record lengths, lack of precision in the measurements, nonstationarity, and, above all, nonindependence (or persistence) in the data all cast doubt on the validity of the results. Of these factors, perhaps the most pernicious (because so often unrecognized) is persistence.

A technique commonly used by communications engineers to enhance signal-to-noise ratios may be of benefit here. This is the filtering of the data to remove power at unwanted frequencies. In its simplest form, this consists of subtracting the mean value from each individual data point. The power spectrum of the fluctuations about the mean is then compared with the power spectrum of white noise (a white noise spectrum is of uniform power at all frequencies and results when the random fluctuations are completely independent). In dealing with meteorological data, a further modification should be made to eliminate the effect of persistence on the power spectrum. Specifically, the filter suggested here is

$$Z_t = X_t - \rho_1 X_{t-1} \tag{1}$$

where Z_t is the modified value, X_t and X_{t-1} are the original deviations from the mean at times t and $t-1$, and ρ_1 is the autocorrelation coefficient between X_t and X_{t-1}. (See table 1.) The reasons for choosing this particular filter and its effect on the power spectrum are detailed in the following sections.

TABLE 1.—*Mathematical symbols used*

Y_t	original data at time $t=1, 2, \ldots, N$
X_t	original data, with the mean subtracted
Z_t	modified data after filtering of X_t
N	number of data points available
M	maximum lag used
ρ_n	autocorrelation coefficient of the X_t's at lag n
r_n	autocorrelation coefficient of the Z_t's at lag n
σ_x^2	variance of the X_t's
σ_z^2	variance of the Z_t's
S_p	spectral distribution function of X
P_p	spectral distribution function of Z
T_p	general spectral distribution function
L_p	spectral distribution function of red noise
i	$\sqrt{-1}$
e	base of the natural logarithms
p, q, n	indices to indicate lags

2. MARKOV PROCESSES AND RED NOISE

A simple Markov process is defined as one in which the value at any time depends only on the value of the variable one time step before. If the X's of eq (1) were the result of a simple Markov process and ρ_1 were computed exactly, the Z's would all be identically zero. The autocorrelation coefficient of a simple Markov process decreases exponentially with increasing lag. Thus, the autocorrelation coefficient at lag n equals $(\rho_1)^n$. This can be demonstrated as follows. For a Markov process,

$$X_t = \rho_1 X_{t-1} + W \tag{2}$$

where W is a random variable with zero mean. Assuming that the X's have been expressed as deviations from a mean value, the autocorrelation coefficient at lag 1 is

$$\rho_1 = \overline{X_t X_{t-1}} / \sigma_x^2 \tag{3}$$

while that at lag 2 is

$$\rho_2 = \overline{X_t X_{t-2}} / \sigma_x^2 = \rho_1 \overline{X_{t-1} X_{t-2}} / \sigma_x^2 = \rho_1^2. \tag{4}$$

Similar substitution for autocorrelations at higher lags yields the general result

$$\rho_n = (\rho_1)^n. \tag{5}$$

The power spectrum resulting from a Markov process has a peak at low frequencies and has been dubbed the "power spectrum of red noise," analogous to that of white noise. It has been suggested (Gilman et al. 1963, Mitchell 1964, Ackerman 1967) that those analyzing meteorological data should compare their observed spectrum with that of red noise to determine the significance of any apparent periodicities. Comparing the spectra by eye is not entirely satisfactory; nor is successive fitting of various red noise spectra to the observations. Filtering the data by means of eq (1) and examining the spectrum of the residuals gives more objective, reproducible results.

3. MATHEMATICAL DISCUSSION

Calculation of autocorrelation coefficients and power spectra on digital computers when the data have been sampled at discrete, equal time intervals is well documented (e.g., Southworth 1960). Assume a series of discrete data points Y_t with mean equal $\overline{Y}$. As a first step in calculating the power spectrum, it is customary to form a new data set by subtracting the mean from each data point. Thus,

$$X_t=Y_t-\overline{Y}, \tag{6}$$

$$\overline{X}=0,$$

and

$$\sigma_x^2=\overline{X^2}. \tag{7}$$

It has been shown (e.g., Miller 1956, p. 171 ff.) that the power density function equals the Fourier transform of the autocorrelation coefficient (this equivalence is sometimes referred to as the Wiener-Khintchine Relation). The autocorrelation coefficient at lag n is

$$\rho_n=\overline{X_tX_{t-n}}/\overline{X^2}=\frac{\overline{X_tX_{t-n}}}{\sigma_x^2}. \tag{8}$$

For discrete data, the Fourier transform must be expressed as a finite Fourier series. The spectral density at lag p is then

$$T_p=\rho_0+2\sum_{q=1}^{M-1}\rho_q\cos\frac{qp\pi}{M}+\rho_M\cos p\pi \tag{9}$$

where the ρ's are defined in eq (8). The ρ_0 is the coefficient at zero lag and is identically equal to 1; ρ_M is the autocorrelation coefficient at the maximum lag M. The maximum number of lags used depends on the record length and is generally chosen to be no more than 1/10 the number of data points available.

If the data are the result of a simple Markov process, we can express the autocorrelation coefficients as powers of the autocorrelation/coefficient at lag 1 as shown in eq (5). Substituting this into eq(9) yields

$$L_p=1+2\sum_{q=1}^{M-1}\rho_1^q\cos\frac{qp\pi}{M}+\rho_1^M\cos p\pi. \tag{10}$$

Next, we make the transformation

$$\cos\frac{qp\pi}{M}=1/2(e^{iqp\pi/M}+e^{-iqp\pi/M}). \tag{11}$$

Substituting this into eq (10), we obtain

$$L_p=1+\sum_{q=1}^{M-1}\rho_1^qe^{iqp\pi/M}+\sum_{q=1}^{M-1}\rho_1^qe^{-iqp\pi/M}+\rho_1^M\cos p\pi. \tag{12}$$

The second and third terms of this equation are the sums of geometric progressions and can be evaluated. Equation (12) then becomes

$$L_p=1+\frac{\rho_1^Me^{ip\pi}-\rho_1e^{ip\pi/M}}{\rho_1e^{ip\pi/M}-1}+\frac{\rho_1^Me^{-ip\pi}-\rho_1e^{-ip\pi/M}}{\rho_1e^{-ip\pi/M}-1}+\rho_1^M\cos p\pi. \tag{13}$$

The common denominator of eq (13) is $\rho_1^2-\rho_1(e^{-ip\pi/M}+e^{-ip\pi/M})+1$ and equals $\rho_1^2-2\rho_1\cos(p\pi/M)+1$. Placing the entire expression over this common denominator, we obtain

$$\begin{aligned}L_p\left(1+\rho_1^2-2\rho_1\cos\frac{p\pi}{M}\right)=&\left[\rho_1^2-2\rho_1\cos\frac{p\pi}{M}+1\right]\\&+[\rho_1\rho_1^Me^{[ip\pi-(ip\pi/M)]}-\rho_1^2-\rho_1^Me^{ip\pi}+\rho_1e^{ip\pi/M}]\\&+[\rho_1\rho_1^Me^{-[ip\pi-(ip\pi/M)]}-\rho_1^2-\rho_1^Me^{-ip\pi}+\rho_1e^{-ip\pi/M}]\\&+[\rho_1^M\rho_1^2\cos p\pi-2\rho_1\rho_1^M\cos\frac{p\pi}{M}\cos p\pi+\rho_1^M\cos p\pi].\end{aligned} \tag{14}$$

Collecting terms and reconverting to cosine form whenever possible, we can simplify eq (14) to

$$L_p=\frac{(1-\rho_1^2)(1-\rho_1^M\cos p\pi)+2\rho_1\rho_1^M\left[\cos\left(p\pi-\frac{p\pi}{M}\right)-\cos\frac{p\pi}{M}\cos p\pi\right]}{1+\rho_1^2-2\rho_1\cos\frac{p\pi}{M}}. \tag{15}$$

Since p is always an integer, $\sin p\pi=0$, and $\cos p\pi=(-1)^p$. Making these substitutions and applying the formula for $\cos(a-b)$, we obtain

$$L_p=\frac{(1-\rho_1^2)(1-(-1)^p\rho_1^M)}{1+\rho_1^2-2\rho_1\cos\frac{p\pi}{M}}. \tag{16}$$

By definition, ρ_1 is always less than 1, and M is large enough so L_p approaches the limiting form

$$L_p=\frac{1-\rho_1^2}{1+\rho_1^2-2\rho_1\cos\frac{p\pi}{M}}. \tag{17}$$

This is the formula for the power spectrum of red noise given by Gilman et al. (1963).

Now consider the variable Z_t defined in eq (1). Simple substitution gives

$$\overline{Z}=(1-\rho_1)\overline{X}=0 \tag{18}$$

and

$$\sigma_z^2=\overline{Z^2}=(1-\rho_1^2)\sigma_x^2. \tag{19}$$

Let S_p denote the power density function of the X's; r_n, the autocorrelation coefficient of the Z's at lag n; and P_p, their power density function. Then,

$$P_p=r_0+2\sum_{q=1}^{M-1}r_q\cos\frac{pq\pi}{M}+r_M\cos p\pi, \tag{20}$$

$$r_0=\rho_0=1, \tag{21}$$

$$r_1=\frac{\overline{Z_tZ_{t-1}}}{\sigma_z^2}=\frac{(1+\rho_1^2)\rho_1\sigma_x^2-\rho_1\sigma_x^2-\rho_1\rho_2\sigma_x^2}{(1-\rho_1^2)\sigma_x^2}=\frac{(\rho_1^2-\rho_2)\rho_1}{(1-\rho_1^2)}$$

$$=r_1=\frac{(1+\rho_1^2)\rho_1-\rho_1(\rho_0+\rho_2)}{1-\rho_1^2}, \tag{22}$$

$$r_2=\frac{(1+\rho_1^2)\rho_2-\rho_1(\rho_1+\rho_3)}{1-\rho_1^2}; \tag{23}$$

or, in general,

$$r_n=\frac{(1+\rho_1^2)\rho_n-\rho_1(\rho_{n-1}+\rho_{n+1})}{1-\rho_1^2}. \tag{24}$$

Therefore, eq (20) can be rewritten as

$$P_p=\frac{1}{1-\rho_1^2}\left[1-\rho_1^2+2\sum_{q=1}^{M-1}\{(1+\rho_1^2)\rho_q-\rho_1(\rho_{q-1}+\rho_{q+1})\}\right.$$
$$\left.\times\cos\frac{pq\pi}{M}+[(1+\rho_1^2)\rho_M-\rho_1(\rho_{M-1}+\rho_{M+1})]\cos p\pi\right]. \tag{25}$$

Collecting terms and making use of eq (9), we obtain

$$(1-\rho_1^2)P_p=S_p(1+\rho_1^2)-2\rho_1\left[\rho_1+\sum_{q=1}^{M-1}(\rho_{q-1}+\rho_{q+1})\cos\frac{pq\pi}{M}\right]$$
$$-\rho_1(\rho_{M-1}+\rho_{M+1})\cos p\pi. \tag{26}$$

In eq (26), the term inside the summation sign can be rearranged by collecting terms in ρ_q, and becomes

$$\sum_{q=1}^{M-1}\rho_q\left[\cos\frac{(q+1)p\pi}{M}+\cos\frac{(q-1)p\pi}{M}\right]$$
$$+\frac{\cos p\pi}{M}-\rho_1-\rho_{M-1}\cos p\pi+\rho_M\cos\frac{(M-1)p\pi}{M}. \tag{27}$$

Placing this result into eq (26), we obtain

$$(1-\rho_1^2)P_p=S_p(1+\rho_1^2)-2\rho_1\left[\cos\frac{p\pi}{M}+\rho_M\cos\frac{p\pi}{M}\cos p\pi\right.$$
$$\left.+\sum_{q=1}^{M-1}\rho_q\left(\cos\frac{(q+1)p\pi}{M}+\cos\frac{(q-1)p\pi}{M}\right)\right]$$
$$+\rho_1(\rho_{M-1}-\rho_{M+1})\cos p\pi. \tag{28}$$

Since $\cos A+\cos B=2\cos(1/2)(A+B)\cos(1/2)(A-B)$, eq (28) becomes

$$(1-\rho_1^2)\,P_p=S_p(1+\rho_1^2)-2\rho_1\cos\frac{p\pi}{M}\left[1+\rho_M\cos p\pi\right.$$
$$\left.+2\sum_{q-1}^{M-1}\rho_q\cos\frac{qp\pi}{M}\right]+\rho_1\,(\rho_{M-1}-\rho_{M+1})\cos p\pi. \tag{29}$$

The last term in eq (29) approaches zero for large M, giving

$$P_p=S_p\left[\frac{1+\rho_1^2-2\rho_1\cos\frac{p\pi}{M}}{1-\rho_1^2}\right] \tag{30}$$

as the limiting case. Blackman and Tukey (1958, pp. 41 and 126) discuss similar procedures without giving details.

The modification of the data according to eq (1) produces a spectrum that equals the spectrum of the original data times the reciprocal of a red noise spectrum (eq 17). This verifies the contention that the filtering introduced by eq (1) eliminates that portion of the spectrum due to a simple Markov process.

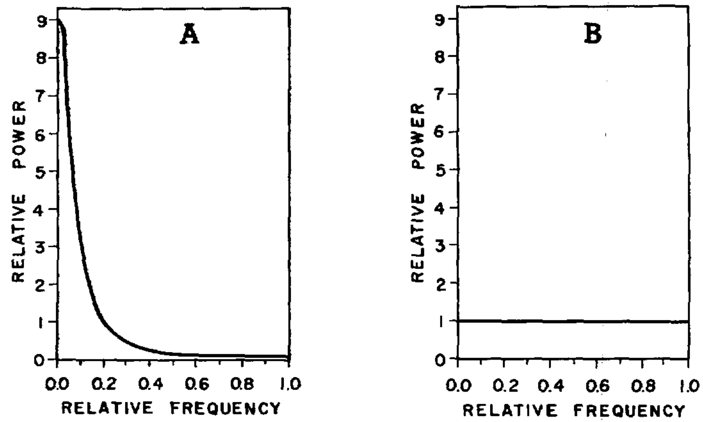

FIGURE 1.—(A) the spectrum that results from a Markov process with $\rho_1=0.8$ and (B) the white noise spectrum that results when the data with spectrum (A) are filtered by eq (1).

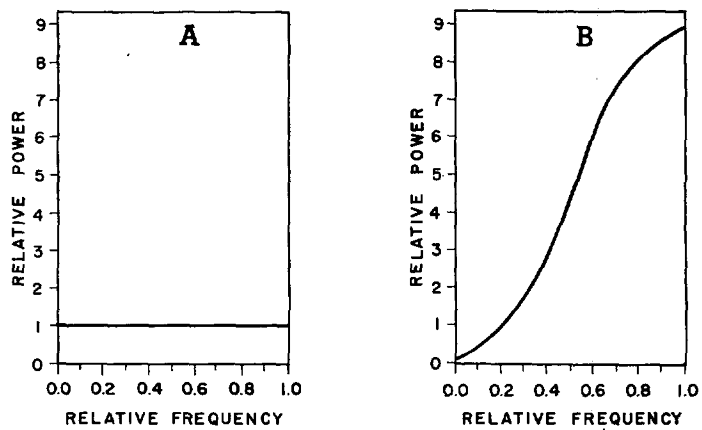

FIGURE 2.—(A) the white noise spectrum resulting from a series of independent random variables and (B) the spectrum that results when independent random variables are filtered by eq (1), setting $\rho_1=0.8$.

4. EXAMPLES OF THE FILTER'S EFFECTS ON POWER SPECTRA

Suppose the original data (the X's) were the result of a simple Markov process, with a spectrum defined by eq (17). If the value of ρ_1 were 0.8, the original power spectrum would look like that in figure 1A. The spectrum of the modified data (the Z's) would be simply a white noise spectrum (fig. 1B). This example itself is trivial; a purely Markovian process would produce Z's identically equal to zero. Not quite so trivial is the conclusion that the more nearly the spectrum of the modified data approaches the spectrum of white noise, the more closely the original data resembled the output of a simple Markov process.

Let us now consider original data that were completely random, producing a white noise spectrum as in figure 2A. In practice, the filtering of eq (1) would never take place because the autocorrelation coefficients at successive time intervals would be discouragingly low. However, if the

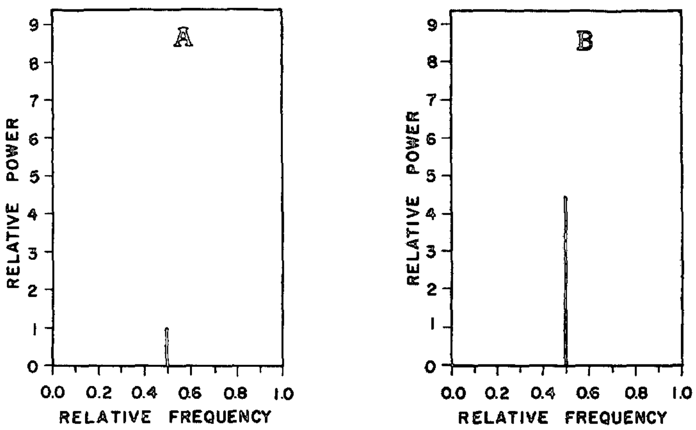

FIGURE 3.—(A) the spectrum that results from a purely periodic function with resonant frequency one-half the Nyquist frequency ($f_n=1/2t$) and (B) the magnification of spectrum (A) after the data are modified according to eq (1), with $\rho_1=0.8$. At other frequencies the amplitude may be reduced, rather than increased, by filtering.

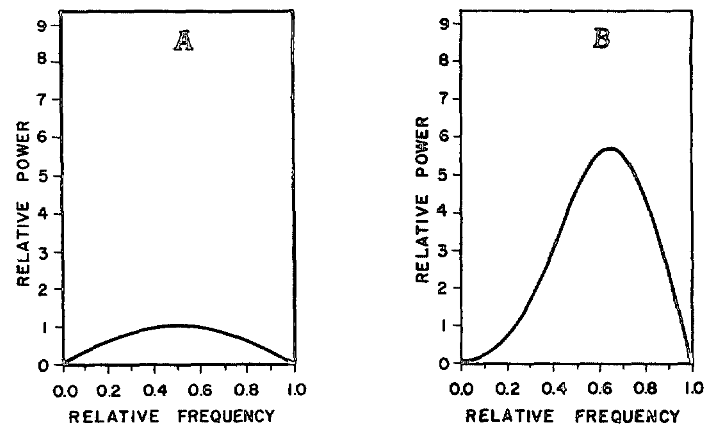

FIGURE 4.—(A) the spectrum with maximum amplitude at frequency$=0.5\,f_n$ and (B) the spectrum that results from modifying the data of (A) by eq (1) and with $\rho_1=0.8$. Note that the maximum amplitude has shifted to $f=0.6\,f_n$.

data were modified (again setting $\rho_1=0.8$), the resulting spectrum would be as shown in figure 2B. This is also the output of the filter defined in eq (1). A spectrum such as this would be a warning that the filter should not have been used in this instance. If the spectrum of the original data does not resemble that of white noise and if a spectrum similar to figure 2B still occurs, then one can assume that a frequency component very close to the folding frequency is present.

If the original data has a pronounced periodicity such that $S_p=1$ for $p=M/2$ and zero elsewhere, the spectrum would be a line spectrum, as shown in figure 3A. The spectrum of the modified data (with $\rho_1=0.8$) would be that of figure 3B. Thus, pure periodicity is not lost by filtering. In this instance, the peak was even magnified. At other frequencies or with a different choice of ρ_1, this may not be the case. Original data that consisted of combinations of characteristics illustrated in figures 1A and 3A would after filtering produce spectra that were modifications of the characteristics shown in figures 1B and 3B.

Suppose there are several periods in the original data or, at least, no single line spectrum. What would the filtering process do then? Figure 4A is an example of a sinusoidal spectrum with a peak at the center of the frequency range. We will not discuss here the problems that arise in interpreting the results of the analysis of discrete data (viz, resolution and aliasing) but will assume that figure 4A gives the true spectrum of the original data. Figure 4B, the spectrum of the modified data (still keeping $\rho_1=0.8$), shows that the peak is larger and occurs at a higher frequency than the original—offhand, not a very good result. The point to remember here is that, if the spectrum of figure 4A were imbedded in red noise, we probably would not be able to recognize that there was even a periodic function present, much less determine its frequency. On the other hand, care should be taken in interpreting the significance of the frequencies observed after filtering.

5. SUMMARY

This paper has attempted to sketch the rationale in proposing that meteorological data be modified to eliminate the effect of persistence. In particular, filtering by means of eq (1) has been suggested; and the results of such filtering under different circumstances have been indicated. The method has already been used in the analysis of time variations in snowfall rates (Dyer 1970), and it is hoped that it can provide a nonsubjective technique of testing for the presence of periodicities in meteorological data.

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